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Some Problems of Best Approximation with Constraints

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We discuss problems of best approximation with constraints in (a) an abstract Hilbert space setting and (b) a concrete form involving polynomial approximation. One problem is to compute the Hilbert space distance from a fixed vector h to the set of vectors Ad such that $\|Bd\| \leqslant M$, where A, B are given linear operators and M is a positive constant. A related concrete problem is to find the $L^2(\mu)$-distance from a fixed function h to the set of polynomials p that satisfy $\int |p|^2 \, dv \leqslant M^2$, where μ, v are nonnegative, finite Borel measures on the unit circle and M is a positive constant. In particular, the dependence of this distance on the singular components of μ and v is investigated.

1. INTRODUCTION

A classical problem is to approximate a given vector h in a Hilbert space by vectors in a linear manifold. In another view, one can think of h as data to be extrapolated, and approximants as the result of extrapolation. The imposition of constraints on approximants is sometimes helpful in making

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the extrapolation process stable [4, 22]. Problems can be posed in abstract or concrete settings; both types are considered in this paper.

Let $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{K}$ be Hilbert spaces, and let A, B be closed, densely defined, linear operators on $\mathbf{K}$ to $\mathbf{H}_1$, $\mathbf{H}_2$, respectively. Let h be a fixed vector in $\mathbf{H}_1$, and let M, λ be positive constants. Set

$$I_M := \inf\{\|h - Ad\|^2 \colon d \in D(A) \cap D(B),\ \|Bd\|^2 \leqslant M^2\}, \tag{1.1}$$

$$\begin{aligned} I_\infty &:= \inf\{\|h - Ad\|^2 \colon d \in D(A)\} \\ &= \|h - h_A\|^2; \end{aligned} \tag{1.2}$$

here h_A is the projection of h on $(R(A))^-$. We also consider

$$J_\lambda := \inf\{\|h - Ad\|^2 + \lambda\|Bd\|^2 \colon d \in D(A) \cap D(B)\}, \tag{1.3}$$

$$\begin{aligned} J_0 &:= \inf\{\|h - Ad\|^2 \colon d \in D(A)\} \\ &= \|h - h_A\|^2. \end{aligned} \tag{1.4}$$

Appropriate technical conditions are specified in Section 2. For $M \to \infty$ and $\lambda \downarrow 0$,

$$I_M \downarrow I_\infty \qquad \text{and} \qquad J_\lambda \downarrow J_0.$$

It will be shown that I_M always has an extremal vector e_M, and e_M is unique if $I_M > I_\infty$. The infimum J_λ has a unique extremal f_λ. If M and λ are suitably connected, then $e_M = f_\lambda$. We exhibit the required connection and compute the extremal vectors (Section 2). Special cases of these results are given in Rosenblum [18], Shapiro [22], and elsewhere in the literature such as [2, 13]. The results are abstract generalizations of a method of Davis [5].

In Sections 3–5 we take up related problems of polynomial approximation. Let μ, ν be nonnegative, finite Borel measures on the unit circle $\Gamma = \{\zeta \colon |\zeta| = 1\}$. Let h be a given function in $L^2(\mu)$, and let M, λ be positive constants. Set

$$S_M(\mu, \nu) := \inf\left\{\int |h - p|^2\, d\mu \colon p \in \mathscr{P},\ \int |p|^2\, d\nu \leqslant M^2\right\}, \tag{1.5}$$

$$S_\infty(\mu) := \inf\left\{\int |h - p|^2\, d\mu \colon p \in \mathscr{P}\right\} \tag{1.6}$$

and

$$T_\lambda(\mu, \nu) := \inf\left\{\int |h - p|^2\, d\mu + \lambda \int |p|^2\, d\nu \colon p \in \mathscr{P}\right\}, \tag{1.7}$$

$$T_\infty(\mu) := \inf\left\{\int |h - p|^2\, d\mu \colon p \in \mathscr{P}\right\}. \tag{1.8}$$

Under suitable hypotheses, these quantities are independent of the absolutely continuous components of μ and ν. In the case of absolutely continuous measures, the abstract theory of Section 2 is applicable; we interpret the abstract results in this concrete setting.

The unconstrained infimum $S_\infty(\mu) = T_0(\mu)$ is computed in Rosenblum and Widom [20]. The best known case, of course, is $h(\zeta) = \bar{\zeta}$. Then we have the classical Szegö infimum (Grenander and Szegö [8]). Constrained infimum problems of the type that we study were first treated by Kreĭn and Nudel'man [11, 12] in connection with an engineering problem. See also [1, 20]. A different type of problem is obtained by imposing constraints on the degree of a polynomial [23, 24].

Notation

D	open disk in $\mathbf{C}$,
Γ	unit circle (boundary of D),
σ	normalized Lebesgue measure on Γ,
$\mathscr{P}$	set of polynomials,
$\mathbf{H}^p$	Hardy class on D or Γ (as required by context),
$\\|\cdot\\|$	norm,
$\langle \cdot, \cdot \rangle$	inner product,
$D(\cdot)$	domain of an operator,
$R(\cdot)$	range of an operator,
$[\cdot, \cdot]^t$	column vector; t denotes matrix transpose,
$:=$	this indicates a definition.

2. Best Approximation in Hilbert Space

Throughout this section, A, B are closed, densely defined linear operators on a Hilbert space $\mathbf{K}$ to Hilbert spaces $\mathbf{H}_1, \mathbf{H}_2$, respectively. Let h be a fixed vector in $\mathbf{H}_1$, and let h_A be the projection of h on $(R(A))^-$.

Define I_M, I_∞ and J_λ, J_0 by (1.1)–(1.4) for any positive constants M and λ. We adopt the following conditions as standing hypotheses.

(C1) The set $\mathscr{D} := D(A) \cap D(B)$ is dense in $\mathbf{K}$.

(C2) There is a $\delta > 0$ such that $\|Ad\|^2 + \|Bd\|^2 \geqslant \delta \|d\|^2$ for all $d \in \mathscr{D}$.

(C3) The set $A\mathscr{D}$ is dense in $(R(A))^-$.

(C4) The vector h_A is not of the form $h_A = Ad$, where $d \in \mathscr{D}$ and $Bd = 0$.

No use of (C4) is made until Theorem 2.6. Its use there is to eliminate a trivial case: if (C4) fails, then $I_M = I_\infty$ and $J_\lambda = J_0$ for all positive constants M and λ.

2.1. THEOREM. (i) *For any positive constant* M, I_M *has an extremal vector* e_M. *If* e_M^1, e_M^2 *are two extremal vectors for* I_M, *then* $Ae_M^1 = Ae_M^2$.

(ii) *For any positive constant* λ, J_λ *has a unique extremal vector* f_λ.

For now e_M denotes any extremal vector for I_M, the choice being immaterial. We will see later that if $I_M > I_\infty$, then e_M is itself unique. By the definition of I_M, for any $M \in (0, \infty)$,

$$e_M \in \mathscr{D}, \qquad \|Be_M\|^2 \leqslant M^2, \qquad \text{and} \qquad \|h - Ae_M\|^2 = I_M.$$

By the definition of J_λ, for any $\lambda \in (0, \infty)$,

$$f_\lambda \in \mathscr{D} \qquad \text{and} \qquad \|h - Af_\lambda\|^2 + \lambda \|Bf_\lambda\|^2 = J_\lambda.$$

Proof. (i) It is sufficient to show that

$$\mathbf{C}_M := \{Ad\colon d \in \mathscr{D}, \|Bd\|^2 \leqslant M^2\}$$

is a closed convex set in $\mathbf{H}_1$, since once this is known the assertions of (i) follow from well-known properties of closed convex sets in a Hilbert space.

It is clear that $\mathbf{C}_M$ is convex. If a is in the closure of $\mathbf{C}_M$, then $Ad_n \to a$ strongly for some sequence $\{d_n\}_1^\infty$ in $\mathscr{D}$ such that $\|Bd_n\|^2 \leqslant M^2$ for all $n \geqslant 1$. By weak compactness we can assume that $Bd_n \to b$ weakly for some b in $\mathbf{H}_2$. By replacing $\{d_n\}_1^\infty$ by a suitable sequence of convex combinations, we can in fact assume that $Bd_n \to b$ strongly in $\mathbf{H}_2$. By (C2), for all $m, n \geqslant 1$,

$$\delta \|d_m - d_n\|^2 \leqslant \|Ad_m - Ad_n\|^2 + \|Bd_m - Bd_n\|^2.$$

Therefore $d_n \to d$ strongly for some d in $\mathbf{K}$. Since A and B are closed, $d \in D(A) \cap D(B) = \mathscr{D}$, $Ad = a$, and $Bd = b$. It follows that $a \in \mathbf{C}_M$, and so $\mathbf{C}_M$ is closed and (i) follows.

(ii) We view $\mathbf{H}_1 \times \mathbf{H}_2$ as a space of column vectors. Define a closed linear operator $\mathbf{C}_\lambda$ on $\mathbf{K}$ to $\mathbf{H}_1 \times \mathbf{H}_2$ by

$$D(\mathbf{C}_\lambda) = \mathscr{D} \qquad \text{and} \qquad \mathbf{C}_\lambda d = [Ad, \lambda^{1/2} Bd]', \quad d \in \mathscr{D}.$$

For any $d \in \mathscr{D}$,

$$\|h - Ad\|^2 + \lambda \|Bd\|^2 = \|[h, 0]' - \mathbf{C}_\lambda d\|^2.$$

For some sequence $\{d_n\}_1^\infty$ in $\mathscr{D}$,

$$\lim_{n \to \infty} \|[h, 0]' - \mathbf{C}_\lambda d_n\|^2 = J_\lambda.$$

A standard application of the parallelogram law shows that $\mathbf{C}_\lambda d_n \to [u, v]'$ strongly, where $u \in \mathbf{H}_1$, $v \in \mathbf{H}_2$, and $\|h-u\|^2 + \|v\|^2 = J_\lambda$. An application of (C2) as in the proof of (i) shows that $d_n \to f_\lambda$ strongly for some $f_\lambda \in \mathbf{K}$. Since $\mathbf{C}_\lambda$ is closed, $f_\lambda \in \mathscr{D}$ and $\mathbf{C}_\lambda f_\lambda = [u, v]'$. Thus f_λ is an extremal vector for J_λ. The uniqueness of f_λ follows from another application of the parallelogram law and (C2). ∎

2.2. THEOREM. *For* $M \to \infty$, $I_M \downarrow I_\infty$ *and* $\|h_A - Ae_M\| \to 0$. For $\lambda \downarrow 0$, $J_\lambda \downarrow J_0$ *and* $\|h_A - Af_\lambda\| \to 0$.

Proof. By (C3) and the definitions of I_∞ and J_0,

$$I_\infty = J_0 = \inf\{\|h - Ad\|^2 : d \in \mathscr{D}\}.$$

The rest is elementary and left to the reader. ∎

2.3. LEMMA. *Fix* $M \in (0, \infty)$.

(i) *Let* $d \in \mathscr{D}$, *and suppose that there is an* $\varepsilon > 0$ *such that*

$$\|B(\alpha e_M + \beta d)\|^2 \leqslant M^2 \tag{2.1}$$

for all α, $\beta \in \mathbf{C}$ *satisfying* $|\alpha|^2 + |\beta|^2 \leqslant 1$ *and* $|\beta| \leqslant \varepsilon$. *Then* $h - Ae_M \perp Ad$ *in* $\mathbf{H}_1$.

(ii) *If* $d \in \mathscr{D}$ *and* $Be_M \perp Bd$ *in* $\mathbf{H}_2$, *then* $h - Ae_M \perp Ad$ *in* $\mathbf{H}_1$.

(iii) *We have* $\langle h - Ae_M, Ae_M \rangle \geqslant 0$.

Proof. As a preliminary, note that if $d \in \mathscr{D}$ and α, β are any numbers satisfying (2.1), then

$$\begin{aligned}\|h - Ae_M\|^2 &\leqslant \|h - A(\alpha e_M + \beta d)\|^2 \\ &= \|h - Ae_M - A((\alpha - 1)\, e_M + \beta d)\|^2.\end{aligned}$$

Expanding and simplifying, we obtain

$$\begin{aligned}0 \leqslant\ & 2\,\mathrm{Re}(1-\alpha)\langle Ae_M, h - Ae_M\rangle - 2\,\mathrm{Re}\,\beta\langle Ad, h - Ae_M\rangle \\ &+ \|A((\alpha - 1)\, e_M + \beta d)\|^2.\end{aligned} \tag{2.2}$$

To prove (i), in (2.2) set $\alpha = 1 - t$ and $\beta = (2t - t^2)^{1/2} e^{i\theta}$, where t is a small positive number and θ is chosen so that

$$e^{i\theta}\langle Ad, h - Ae_M\rangle = |\langle Ad, h - Ae_M\rangle|.$$

This yields

$$0 \leqslant -2(2t-t^2)^{1/2}\,|\langle Ad, h-Ae_M\rangle| + \mathcal{O}(t)$$

for $t\downarrow 0$. Hence $\langle Ad, h-Ae_M\rangle = 0$, and this proves (i).

We obtain (ii) in a routine way from (i).

To prove (iii), consider any real θ with $\cos\theta > 0$. Set $\alpha = 1 - te^{i\theta}$, where $t > 0$. Then for all sufficiently small t, $|\alpha| < 1$. Applying (2.2) with $\beta = 0$, we get

$$0 \leqslant 2\,\mathrm{Re}\; te^{i\theta}\langle Ae_M, h-Ae_M\rangle + \mathcal{O}(t^2)$$

for $t\downarrow 0$. Hence $\mathrm{Re}\; e^{i\theta}\langle Ae_M, h-Ae_M\rangle \geqslant 0$. By the arbitrariness of θ, (iii) follows. ∎

2.4. LEMMA. *Fix $M\in(0,\infty)$ and assume $I_M > I_\infty$. Then*

(i) $\|Be_M\|^2 = M^2$ *and*

(ii) *there is a unique positive real number $\lambda(M)$ such that*

$$\langle h-Ae_M, Ad\rangle = \lambda(M)\langle Be_M, Bd\rangle \tag{2.3}$$

for all $d\in\mathcal{D}$.

Proof. The inequality $\|Be_M\|^2 \leqslant M^2$ is automatic. If this inequality is strict, then by 2.3(i),

$$h-Ae_M \perp A\mathcal{D} \quad \text{in} \quad \mathbf{H}_1. \tag{2.4}$$

Then by (C3), $h-Ae_M \perp R(A)$, and so $Ae_M = h_A$. However, this implies $I_M = I_\infty$, contrary to assumption. This proves (i).

To prove (ii), consider any $d\in\mathcal{D}$, and set

$$a = \|Be_M\|^2\, d - \langle Bd, Be_M\rangle\, e_M.$$

Then $a\in\mathcal{D}$ and $Be_M \perp Ba$ in $\mathbf{H}_2$. By 2.3(ii),

$$\begin{aligned} 0 &= \langle h-Ae_M, Aa\rangle \\ &= \|Be_M\|^2\,\langle h-Ae_M, Ad\rangle - \langle h-Ae_M, Ae_M\rangle\langle Be_M, Bd\rangle. \end{aligned}$$

Thus (2.3) holds with

$$\begin{aligned} \lambda(M) &= \langle h-Ae_M, Ae_M\rangle / \|Be_M\|^2 \\ &= \langle h-Ae_M, Ae_M\rangle\, M^{-2}. \end{aligned}$$

By 2.3(iii), $\lambda(M) \geqslant 0$. If $\lambda(M)=0$, then by (2.3), (2.4) holds. As above this implies (via (C3)) that $I_M = I_\infty$, a contradiction. So $\lambda(M)>0$. The uniqueness of $\lambda(M)$ is clear. ∎

We next introduce a certain family $\{T_\lambda\}_{\lambda>0}$ of selfadjoint operators on **K**. Formally we would like to set

$$T_\lambda = A^*A + \lambda B^*B$$

for any $\lambda \in (0, \infty)$. The rigorous definition is

$$T_\lambda := \mathbf{C}_\lambda^* \mathbf{C}_\lambda, \tag{2.5}$$

where $\mathbf{C}_\lambda := [A, \lambda^{1/2}B]'$ is as in the proof of Theorem 2.1(ii). Thus $\mathbf{C}_\lambda$ is a closed, densely defined linear operator on **K** to $\mathbf{H}_1 \times \mathbf{H}_2$ viewed as a space of column vectors. It follows that T_λ is a selfadjoint operator (Riesz and Sz.-Nagy [15, p. 312]), and $D(T_\lambda) \subseteq D(\mathbf{C}_\lambda) = \mathscr{D}$. By (C2) the spectrum of T_λ has a positive lower bound. Therefore T_λ^{-1} and $T_\lambda^{-1/2}$ exist as everywhere defined and bounded linear operators on **K**. It is easy to see that the range of $\mathbf{C}_\lambda$, $R(\mathbf{C}_\lambda) = \mathbf{C}_\lambda \mathscr{D}$, is a closed linear manifold in $\mathbf{H}_1 \times \mathbf{H}_2$.

2.5. LEMMA. *Fix $\lambda \in (0, \infty)$.*

(i) *The operators*

$$A_\lambda := A T_\lambda^{-1/2} \qquad \text{and} \qquad B_\lambda := B T_\lambda^{-1/2}$$

are everywhere defined and bounded on **K** *with $\|A_\lambda\| \leqslant 1$ and $\|B_\lambda\| \leqslant \lambda^{-1/2}$. In fact, the operator on* **K** *to $\mathbf{H}_1 \times \mathbf{H}_2$ defined by*

$$W_\lambda := [A_\lambda, \lambda^{1/2} B_\lambda]'$$

maps **K** *isometrically onto $\mathbf{C}_\lambda \mathscr{D}$. Hence*

$$A_\lambda^* A_\lambda + \lambda B_\lambda^* B_\lambda = W_\lambda^* W_\lambda = I_{\mathbf{K}}$$

is the identity operator on **K**, *and*

$$\begin{bmatrix} A_\lambda A_\lambda^* & \lambda^{1/2} A_\lambda B_\lambda^* \\ \lambda^{1/2} B_\lambda A_\lambda^* & \lambda B_\lambda B_\lambda^* \end{bmatrix} = W_\lambda W_\lambda^*$$

is the projection of $\mathbf{H}_1 \times \mathbf{H}_2$ onto $\mathbf{C}_\lambda \mathscr{D}$.

(ii) *The operators $A T_\lambda^{-1}$ and $B(A T_\lambda^{-1})^*$ are everywhere defined and bounded on* **K** *and $\mathbf{H}_1$, respectively.*

(iii) *For each $d \in \mathscr{D}$ there exist vectors $\{d_\varepsilon\}_{\varepsilon>0}$ in $D(T_\lambda)$ such that $d_\varepsilon \to d$ strongly in* **K** *and $\mathbf{C}_\lambda d_\varepsilon \to \mathbf{C}_\lambda d$ strongly in $\mathbf{H}_1 \times \mathbf{H}_2$ as $\varepsilon \downarrow 0$.*

Proof. Let $\mathbf{C}_\lambda = V\mathbf{H}$ be the polar decomposition of $\mathbf{C}_\lambda$ (von Neumann [14]). Then $\mathbf{H} = (\mathbf{C}_\lambda^* \mathbf{C}_\lambda)^{1/2} = T_\lambda^{1/2}$. Hence $D(\mathbf{H}) = D(T_\lambda^{1/2}) = D(\mathbf{C}_\lambda) = \mathscr{D}$. The operator V is an isometry on $\mathbf{K}$ to $\mathbf{H}_1 \times \mathbf{H}_2$ with range $\mathbf{C}_\lambda \mathscr{D}$. Since $T_\lambda^{-1/2}$ exists as an everywhere defined and bounded operator on $\mathbf{K}$,

$$V = \mathbf{C}_\lambda T_\lambda^{-1/2} = [A_\lambda, \lambda^{1/2} B_\lambda]'.$$

In other words, $V = W_\lambda$. The assertions of (i) now follow.

It is easy to deduce (ii) from (i).

To prove (iii), given $d \in \mathscr{D}$ set

$$d_\varepsilon = (I + \varepsilon T_\lambda)^{-1}\, d, \qquad \varepsilon > 0.$$

Then $d_\varepsilon \in D(T_\lambda)$ for all $\varepsilon > 0$. In a straightforward way we obtain $d_\varepsilon \to d$ strongly in $\mathbf{K}$, and

$$\mathbf{C}_\lambda\, d_\varepsilon = W_\lambda (I + \varepsilon T_\lambda)^{-1}\, T_\lambda^{1/2}\, d \to W_\lambda T_\lambda^{1/2}\, d = \mathbf{C}_\lambda\, d$$

strongly in $\mathbf{H}_1 \times \mathbf{H}_2$ as $\varepsilon \downarrow 0$. ∎

We are now ready to state and prove the main result of this section.

2.6. Theorem. *Define*

$$m(t) := \|B(AT_t^{-1})^*\, h\|^2, \qquad 0 < t < \infty, \tag{2.6}$$

$$m(0) := \sup_{0 < t < \infty} m(t) \qquad (\leqslant \infty). \tag{2.7}$$

Then $m(\cdot)$ is a positive, strictly decreasing, continuous function on $(0, \infty)$ with limit 0 at ∞. Let M be a positive constant.

(i) *If $M^2 \geqslant m(0)$, then $I_M = I_\infty$.*

(ii) *If $M^2 < m(0)$, then $I_M > I_\infty$. In this case, $M^2 = m(\lambda)$ for a unique $\lambda \in (0, \infty)$. Then I_M and J_λ have the unique extremal vector*

$$e_M = f_\lambda = (AT_\lambda^{-1})^*\, h \tag{2.8}$$

and

$$\lambda M^2 + I_M = J_\lambda. \tag{2.9}$$

The condition (C4) will put in its first appearance in the proof of Theorem 2.6. It is not used in either of the following two lemmas.

2.7. Lemma. *For any $t \in (0, \infty)$,*

$$B(AT_t^{-1})^* = \lim_{\varepsilon \downarrow 0} B_\varepsilon (I + tB_\varepsilon^* B_\varepsilon)^{-1}\, A_\varepsilon^* \tag{2.10}$$

in the operator norm.

2.8. LEMMA. *Define $m(t)$ for $t \in (0, \infty)$ by (2.6). There is a nonnegative, finite Borel measure ρ on $[0, \infty)$ such that $\rho([0, \infty)) \leqslant \|h\|^2$ and*

$$m(t) = \int_{[0,\infty)} s^2(1 + ts^2)^{-1}\, d\rho(s) \tag{2.11}$$

for all $t \in (0, \infty)$.

Proof of 2.7. We first show that for any $\varepsilon > 0$,

$$T_t^{-1} - T_{t+\varepsilon}^{-1} = \varepsilon T_t^{-1/2} B_t^* B_{t+\varepsilon} T_{t+\varepsilon}^{-1/2} \tag{2.12}$$

and

$$T_{t+\varepsilon}^{-1} = T_\varepsilon^{-1/2}(I + tB_\varepsilon^* B_\varepsilon)^{-1}\, T_\varepsilon^{-1/2}. \tag{2.13}$$

All of the operators appearing in (2.12) and (2.13) are everywhere defined and bounded. It is enough to check (2.12) on a dense set; a convenient choice for this is $R(T_{t+\varepsilon})$. Thus the proof of (2.12) reduces to showing that

$$\langle T_t^{-1} T_{t+\varepsilon} u, g\rangle - \langle u, g\rangle = \varepsilon \langle T_t^{-1/2} B_t^* B u, g\rangle$$

for all $u \in D(T_{t+\varepsilon})$ and a dense set of g's in $\mathbf{K}$. The identity is easily verified for $g \in R(T_t)$, and so (2.12) follows. We similarly reduce (2.13) to showing that

$$\langle (I + tB_\varepsilon^* B_\varepsilon)\, T_\varepsilon^{1/2} u, g\rangle = \langle T_\varepsilon^{-1/2} T_{t+\varepsilon} u, g\rangle$$

for all $u \in D(T_{t+\varepsilon})$ and a dense set of g's in $\mathbf{K}$. A convenient choice now is $g \in R(T_\varepsilon^{1/2})$. The details are straightforward, and (2.13) follows.

By (2.13),

$$\begin{aligned} B(AT_{t+\varepsilon}^{-1})^* &= B(AT_\varepsilon^{-1/2}(I + tB_\varepsilon^* B_\varepsilon)^{-1}\, T_\varepsilon^{-1/2})^* \\ &= B_\varepsilon(I + tB_\varepsilon^* B_\varepsilon)^{-1}\, A_\varepsilon^*. \end{aligned}$$

Hence by (2.12),

$$\begin{aligned} B(AT_t^{-1})^* - B_\varepsilon(I + tB_\varepsilon^* B_\varepsilon)^{-1}\, A_\varepsilon^* &= B(A(T_t^{-1} - T_{t+\varepsilon}^{-1}))^* \\ &= B(AT_\varepsilon^{-1/2} B_t^* B_{t+\varepsilon} T_{t+\varepsilon}^{-1/2})^* \\ &= \varepsilon B_{t+\varepsilon} B_{t+\varepsilon}^* B_t A_t^*. \end{aligned}$$

We obtain (2.10) from this, because by 2.5(i), $\|A_t\| \leqslant 1$, $\|B_t\| \leqslant t^{-1/2}$, $\|B_{t+\varepsilon}\| \leqslant (t+\varepsilon)^{-1/2}$. ∎

Proof of 2.8. By Lemma 2.7, for all $t \in (0, \infty)$,

$$m(t) = \|B(AT_t^{-1})^* h\|^2$$
$$= \lim_{\varepsilon \downarrow 0} \|B_\varepsilon(I + tB_\varepsilon^* B_\varepsilon)^{-1} A_\varepsilon^* h\|^2.$$

Let $B_\varepsilon = V_\varepsilon \mathbf{H}_\varepsilon$ be the polar decomposition of B_ε. Let

$$\mathbf{H}_\varepsilon = \int_{[0,\infty)} s\, dE_\varepsilon(s)$$

be the spectral representation of $\mathbf{H}_\varepsilon$. The spectral measure $E_\varepsilon(\cdot)$ is compactly supported, but we make no use of this fact. Thus,

$$m(t) = \lim_{\varepsilon \downarrow 0} \|\mathbf{H}_\varepsilon(I + t\mathbf{H}_\varepsilon^2)^{-1} A_\varepsilon^* h\|^2$$

$$= \lim_{\varepsilon \downarrow 0} \int_{[0,\infty)} s^2(1 + ts^2)^{-1}\, d\rho_\varepsilon(s),$$

where

$$\rho_\varepsilon(\cdot) = \langle E_\varepsilon(\cdot)\, A_\varepsilon^* h, A_\varepsilon^* h\rangle$$

is a nonnegative Borel measure on $[0, \infty)$ with

$$\rho_\varepsilon([0, \infty)) = \|A_\varepsilon^* h\|^2 \leqslant \|h\|^2$$

by 2.5(i). The existence of a measure ρ having the required properties now follows from a routine compactness argument. ∎

Proof of Theorem 2.6. The heart of the argument is in the proof of assertion (A) below.

(A) Let $0 < M < \infty$ be given, and assume $I_M > I_\infty$. Let $\lambda = \lambda(M)$ be the positive constant of Lemma 2.4. Then the extremal vector e_M for I_M is unique, $M^2 = m(\lambda)$, and (2.8) and (2.9) hold.

Proof of (A). Consider any $u \in \mathbf{K}$. We apply (2.3) with $d := T_\lambda^{-1} u$. We have $d \in D(T_\lambda) \subseteq \mathscr{D}$, and so

$$\langle h, AT_\lambda^{-1} u\rangle = \langle Ae_M, AT_\lambda^{-1} u\rangle + \lambda \langle Be_M, BT_\lambda^{-1} u\rangle$$
$$= \langle \mathbf{C}_\lambda e_M, \mathbf{C}_\lambda T_\lambda^{-1} u\rangle = \langle e_M, u\rangle.$$

The last equality is by (2.5). Therefore e_M is unique and $e_M = (AT_\lambda^{-1})^* h$.

We show that e_M is also extremal for J_λ, that is, the choice $d=e_M$ minimizes

$$\|h-Ad\|^2+\lambda\|Bd\|^2=\|[h,0]'-\mathbf{C}_\lambda d\|^2$$

among all vectors d in $\mathscr{D}$. A sufficient condition for this is that

$$[h,0]'-\mathbf{C}_\lambda e_M \perp \mathbf{C}_\lambda \mathscr{D}. \tag{2.14}$$

By 2.5(iii) it is enough to prove that

$$[h,0]'-\mathbf{C}_\lambda e_M \perp \mathbf{C}_\lambda d$$

for any $d\in D(T_\lambda)$. If $d\in D(T_\lambda)$, then

$$\begin{aligned}\langle [h,0]'-\mathbf{C}_\lambda e_M, \mathbf{C}_\lambda d\rangle &= \langle h, Ad\rangle - \langle e_M, T_\lambda d\rangle\\ &= \langle h, Ad\rangle - \langle h, (AT_\lambda^{-1})\, T_\lambda d\rangle\\ &=0.\end{aligned}$$

Therefore (2.14) holds, and we have shown that e_M is extremal for J_λ. Since J_λ has the unique extremal f_λ by Theorem 2.1(ii), (2.8) holds.

The rest is immediate. By Lemma 2.4(i), $M^2=m(\lambda)$. To prove (2.9), evaluate I_M and J_λ by setting $d=e_M$ in (1.1) and $d=f_\lambda$ in (1.3). This completes the proof of (A).

We finish the proof of Theorem 2.6. By (C4) there is at least one $M\in(0,\infty)$ such that $I_M>I_\infty$. By (A) there is a positive constant $\lambda(M)$ such that $m(\lambda(M))=M^2>0$. Therefore the measure ρ in Lemma 2.8 is not trivial: $\rho((0,\infty))>0$. Hence by the representation (2.11), $m(\cdot)$ is a positive, strictly decreasing, continuous function on $(0,\infty)$ with limit 0 at ∞.

Define $M_0:=\sup\{M\colon I_M>I_\infty\}$. In view of (A), we will be done if we can show:

(B) $M_0^2=m(0)$.

Proof of (B). By (A), the positive constant $\lambda(M)$ of Lemma 2.4 satisfies

$$M^2=m(\lambda(M)) \tag{2.15}$$

for $0<M<M_0$. As $M\uparrow M_0$, $\|h-Ae_M\|\to 0$ (if $M_0=\infty$ this follows from Theorem 2.2; the case $M_0<\infty$ is handled by a separate argument). Hence as $M\uparrow M_0$,

$$\lambda(M)=\langle h-Ae_M, Ae_M\rangle\, M^{-2}\to 0.$$

Therefore we obtain $M_0^2=m(0)$ on letting $M\uparrow M_0$ in (2.15). This completes the proof of (B) and, with (B), the theorem. ∎

3. Polynomial Approximation: Reduction to the Absolutely Continuous Case

Let μ, ν be nonnegative, finite Borel measures on Γ, and let h be a given function in $L^2(\mu)$. Define $S_M(\mu,\nu)$, $S_\infty(\mu)$, and $T_\lambda(\mu, \nu)$, $T_\infty(\mu)$ by (1.5)–(1.8) for any positive constants M and λ. Let

$$\begin{aligned} \mu &= \mu_{ac} + \mu_s, & d\mu_{ac} &= u d\sigma, \\ \nu &= \nu_{ac} + \nu_s, & d\nu_{ac} &= v d\sigma \end{aligned} \tag{3.1}$$

be the Lebesgue decompositions of μ, ν.

It is known that $S_\infty(\mu) = S_\infty(\mu_{ac})$, or, what is the same thing, $T_\infty(\mu) = T_\infty(\mu_{ac})$. See Rosenblum and Widom [19]; the special case $h(\zeta) = \bar{\zeta}$ is a classical theorem of Kolmogorov and Kreĭn (Grenander and Szegö [8]).

3.1. Theorem. *If $\mu_s \perp \nu_s$, then for any positive constants M and λ,*

$$S_M(\mu, \nu) = S_M(\mu_{ac}, \nu_{ac}) \tag{3.2}$$

and

$$T_\lambda(\mu, \nu) = T_\lambda(\mu_{ac}, \nu_{ac}). \tag{3.3}$$

The following lemma isolates the essential content of the theorem.

3.2. Lemma. *Assume $\mu_s \perp \nu_s$. Given $P \in \mathscr{P}$ and $\varepsilon > 0$, there exists $\tilde{P} \in \mathscr{P}$ such that*

$$\int |h - \tilde{P}|^2 \, d\mu < \int |h - P|^2 \, d\mu_{ac} + \varepsilon, \tag{3.4}$$

$$\int |\tilde{P}|^2 \, d\nu < \int |P|^2 \, d\nu_{ac} + \varepsilon. \tag{3.5}$$

Proof of Theorem 3.1 (granting Lemma 3.2). It is clear that for any positive constants M and λ,

$$S_M(\mu, \nu) \geqslant S_M(\mu_{ac}, \nu_{ac}),$$

$$T_\lambda(\mu, \nu) \geqslant T_\lambda(\mu_{ac}, \nu_{ac}).$$

Given $\varepsilon > 0$ we can choose $P \in \mathscr{P}$ so that

$$\int |h - P|^2 \, d\mu_{ac} < S_M(\mu_{ac}, \nu_{ac}) + \varepsilon,$$

$$\int |P|^2 \, d\nu_{ac} \leqslant M^2.$$

By making a small adjustment, we can assume that the last inequality is strict. Then by Lemma 3.2 there exists $\tilde{P} \in \mathscr{P}$ so that

$$\int |h - \tilde{P}|^2 \, d\mu < S_M(\mu_{ac}, \nu_{ac}) + \varepsilon,$$

$$\int |\tilde{P}|^2 \, d\nu < M^2.$$

Hence $S_M(\mu, \nu) < S_M(\mu_{ac}, \nu_{ac}) + \varepsilon$. By the arbitrariness of ε, (3.2) follows.

Similarly, given $\varepsilon > 0$ we can choose $P \in \mathscr{P}$ so that

$$\int |h - P|^2 \, d\mu_{ac} + \lambda \int |P|^2 \, d\nu_{ac} < T_\lambda(\mu_{ac}, \nu_{ac}) + \varepsilon.$$

Then by Lemma 3.2 there exists $\tilde{P} \in \mathscr{P}$ so that

$$\int |h - \tilde{P}|^2 \, d\mu + \lambda \int |\tilde{P}|^2 \, d\nu < T_\lambda(\mu_{ac}, \nu_{ac}) + \varepsilon,$$

and so $T_\lambda(\mu, \nu) < T_\lambda(\mu_{ac}, \nu_{ac}) + \varepsilon$. Since ε is arbitrary, we obtain (3.3). ∎

Proof of Lemma 3.2. Fix $\varepsilon > 0$. We write $\varepsilon_1, \varepsilon_2, \ldots$ for small positive numbers to be chosen in the course of the argument. Set

$$\alpha := \int |h - P|^2 \, d\mu_{ac}, \qquad \beta := \int |P|^2 \, d\nu_{ac}.$$

We proceed in a number of steps.

(i) Choose disjoint, closed subsets E, F of Γ such that $\sigma(E) = \sigma(F) = 0$ and

$$\mu_s(\Gamma \setminus E) < \varepsilon_1 \qquad \text{and} \qquad \nu_s(\Gamma \setminus F) < \varepsilon_2.$$

Such sets exist because μ_s, ν_s are singular measures with $\mu_s \perp \nu_s$, and every finite Borel measure on Γ is regular.

(ii) Let $\mathbf{C}(\Gamma)$ be the set of continuous, complex-valued functions on Γ, and let $\mathscr{A} := \mathbf{H}^\infty \cap \mathbf{C}(\Gamma)$ be the disk algebra. By a theorem of Rudin and Carleson (see Garnett [7, pp. 125–126, Ex. 1d]), there exists $q \in \mathscr{A}$ such that $\|q\|_\infty = 1$ and $q|_E \equiv 1$ and $q|_F \equiv 0$.

(iii) Choose sequences $\{\rho_n\}_1^\infty$ and $\{\tau_n\}_1^\infty$ in $\mathscr{P}$ such that

(a) $|\rho_n| \leqslant 1$ and $|\tau_n| \leqslant 1$ on $\bar{D}$ for all $n \geqslant 1$,

(b) $\lim_{n \to \infty} \int \rho_n \, d\mu_s = \mu_s(\Gamma)$ and $\lim_{n \to \infty} \int \tau_n \, d\nu_s = \nu_s(\Gamma)$, and

(c) $\lim_{n \to \infty} \rho_n(\zeta) = \lim_{n \to \infty} \tau_n(\zeta) = 0$ σ–a.e. on Γ.

Concerning the existence of such polynomials, see Garnett [7, p. 126, Ex. 2].

(iv) Choose $\mathbf{H} \in \mathbf{C}(\Gamma)$ such that $\int |\mathbf{H} - h|^2 \, d\mu < \varepsilon_3$. See Rudin [21, p. 68].

(v) Choose $Q \in \mathscr{A}$ such that $\|Q\|_\infty = \|\mathbf{H}\|_\infty$ and $Q|_E = \mathbf{H}|_E$ (Garnett [7, pp. 125–126, Ex. 1d]).

(vi) Define $\{P_n\}_1^\infty \subseteq \mathscr{A}$ by

$$P_n = P(1-\rho_n)(1-\tau_n) + Q\rho_n q, \qquad n \geqslant 1.$$

For all $n \geqslant 1$,

$$\|P_n\|_\infty \leqslant 4\|P\|_\infty + \|\mathbf{H}\|_\infty.$$

The idea of the proof is to show that for some function $P_n \in \mathscr{A}$ constructed in this way,

$$\int |h - P_n|^2 \, d\mu < \alpha + \varepsilon, \tag{3.6}$$

$$\int |P_n|^2 \, d\nu < \beta + \varepsilon. \tag{3.7}$$

Every function in $\mathscr{A}$ is the uniform limit of the Cesàro means of the partial sums of its Fourier series (Hoffman [9]). Therefore from (3.6), (3.7) we see that there exists a $\tilde{P} \in \mathscr{P}$ satisfying (3.4), (3.5); that is, the result follows from (3.6), (3.7).

We estimate the integrals

$$\mathscr{I} := \int |h - P_n|^2 \, d\mu \qquad \text{and} \qquad \mathscr{J} := \int |P_n|^2 \, d\nu.$$

Put

$$\mathscr{I}_{ac} := \int |h - P_n|^2 \, d\mu_{ac} \qquad \text{and} \qquad \mathscr{J}_{ac} := \int |P_n|^2 \, d\nu_{ac},$$

$$\mathscr{I}_s := \int |h - P_n|^2 \, d\mu_s \qquad \text{and} \qquad \mathscr{J}_s := \int |P_n|^2 \, d\nu_s.$$

We make repeated use of Minkowski's inequality.

To begin,

$$(\mathscr{I}_{ac})^{1/2} \leqslant \left(\int |h - P|^2 \, d\mu_{ac} \right)^{1/2} + \left(\int |P(\rho_n + \tau_n - \rho_n \tau_n) - Q\rho_n q|^2 \, d\mu_{ac} \right)^{1/2}.$$

By (iii) and the Lebesgue dominated convergence theorem, the second term on the right tends to 0 as $n \to \infty$. Therefore

$$(\mathcal{I}_{ac})^{1/2} < \alpha^{1/2} + \varepsilon_4 \tag{3.8}$$

for all sufficiently large n, say $n > n_1$.

For $\mathcal{I}_s$ we have,

$$\begin{aligned}(\mathcal{I}_s)^{1/2} &\leqslant \left(\int |h - Q|^2 \, d\mu_s\right)^{1/2} + \left(\int |Q(1 - \rho_n q)|^2 \, d\mu_s\right)^{1/2} \\ &\quad + \left(\int |P(1 - \tau_n)(1 - \rho_n)|^2 \, d\mu_s\right)^{1/2} \\ &= (\mathcal{I}_{s,1})^{1/2} + (\mathcal{I}_{s,2})^{1/2} + (\mathcal{I}_{s,3})^{1/2},\end{aligned}$$

say. By (v),

$$\begin{aligned}(\mathcal{I}_{s,1})^{1/2} &= \left(\int |(h - \mathbf{H}) + (\mathbf{H} - Q)|^2 \, d\mu_s\right)^{1/2} \\ &\leqslant \left(\int |h - \mathbf{H}|^2 \, d\mu_s\right)^{1/2} + \left(\int_{\Gamma \setminus E} |\mathbf{H} - Q|^2 \, d\mu_s\right)^{1/2}.\end{aligned}$$

Hence by (iv), (v), and (i),

$$(\mathcal{I}_{s,1})^{1/2} < \varepsilon_3^{1/2} + 2\|\mathbf{H}\|_\infty \, \varepsilon_1^{1/2}. \tag{3.9}$$

From (v), (iii), and (ii),

$$\begin{aligned}(\mathcal{I}_{s,2})^{1/2} &= \left(\int |Q(1 - \rho_n) + Q\rho_n(1 - q)|^2 \, d\mu_s\right)^{1/2} \\ &\leqslant \|\mathbf{H}\|_\infty \left[\left(\int |1 - \rho_n|^2 \, d\mu_s\right)^{1/2} + 2\varepsilon_1^{1/2}\right].\end{aligned}$$

By (iii),

$$\begin{aligned}\int |1 - \rho_n|^2 \, d\mu_s &= \int (1 + |\rho_n|^2) \, d\mu_s - 2 \operatorname{Re} \int \rho_n \, d\mu_s \\ &\leqslant 2\left[\mu_s(\Gamma) - \operatorname{Re} \int \rho_n \, d\mu_s\right] \to 0.\end{aligned}$$

Hence $\int |1 - \rho_n|^2 \, d\mu_s < \varepsilon_5$ for all sufficiently large n, say $n > n_2$. So for $n > n_2$,

$$(\mathcal{I}_{s,2})^{1/2} \leqslant \|\mathbf{H}\|_\infty (\varepsilon_5^{1/2} + 2\varepsilon_1^{1/2}). \tag{3.10}$$

Also for $n > n_2$,

$$(\mathscr{I}_{s,3})^{1/2} \leqslant 2\|P\|_\infty \left(\int |1-\rho_n|^2 \, d\mu_s\right)^{1/2} \leqslant 2\|P\|_\infty \varepsilon_5^{1/2}. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we obtain

$$\begin{aligned}(\mathscr{I}_s)^{1/2} &\leqslant (\mathscr{I}_{s,1})^{1/2} + (\mathscr{I}_{s,2})^{1/2} + (\mathscr{I}_{s,3})^{1/2} \\ &< \varepsilon_3^{1/2} + 2\|\mathbf{H}\|_\infty \varepsilon_1^{1/2} + \|\mathbf{H}\|_\infty (\varepsilon_5^{1/2} + 2\varepsilon_1^{1/2}) + 2\|P\|_\infty \varepsilon_5^{1/2} \\ &= 4\|\mathbf{H}\|_\infty \varepsilon_1^{1/2} + \varepsilon_3^{1/2} + (\|\mathbf{H}\|_\infty + 2\|P\|_\infty)\, \varepsilon_5^{1/2} \end{aligned} \tag{3.12}$$

for $n > n_2$. By (3.8) and (3.12),

$$\begin{aligned}\mathscr{I} &= \mathscr{I}_{ac} + \mathscr{I}_s \\ &< (\alpha^{1/2} + \varepsilon_4)^2 + [4\|\mathbf{H}\|_\infty \varepsilon_1^{1/2} + \varepsilon_3^{1/2} + (\|\mathbf{H}\|_\infty + 2\|P\|_\infty)\, \varepsilon_5^{1/2}]^2\end{aligned}$$

for $n > \max(n_1, n_2)$. If ε_1, ε_3, ε_4, ε_5 are small enough, this yields (3.6) for all $n > \max(n_1, n_2)$.

For $\mathscr{I}_{ac}$ we have

$$\begin{aligned}(\mathscr{I}_{ac})^{1/2} &\leqslant \left(\int |P|^2 \, d\nu_{ac}\right)^{1/2} \\ &\quad + \left(\int |P(\rho_n + \tau_n - \rho_n \tau_n) - Q\rho_n q|^2 \, d\nu_{ac}\right)^{1/2}.\end{aligned}$$

By (iii) and the Lebesgue dominated convergence theorem, the second term on the right tends to 0 as $n \to \infty$. Hence

$$(\mathscr{I}_{ac})^{1/2} \leqslant \beta^{1/2} + \varepsilon_6^{1/2} \tag{3.13}$$

for all sufficiently large n, say $n \geqslant n_3$. Also,

$$\begin{aligned}(\mathscr{I}_s)^{1/2} &= \left(\int |P(1-\rho_n)(1-\tau_n) + Q\rho_n q|^2 \, d\nu_s\right)^{1/2} \\ &\leqslant 2\|P\|_\infty \left(\int |1-\tau_n|^2 \, d\nu_s\right)^{1/2} + \|Q\|_\infty \left(\int_{\Gamma \setminus F} d\nu_s\right)^{1/2}.\end{aligned}$$

An argument similar to one given above shows that $\int |1-\tau_n|^2 \, d\nu_s < \varepsilon_7$ for all sufficiently large n, say $n > n_4$. Recalling also (i), we obtain, for $n > n_4$,

$$\begin{aligned}(\mathscr{I}_s)^{1/2} &\leqslant 2\|P\|_\infty \varepsilon_7^{1/2} + \|Q\|_\infty \varepsilon_2^{1/2} \\ &= 2\|P\|_\infty \varepsilon_7^{1/2} + \|\mathbf{H}\|_\infty \varepsilon_2^{1/2}.\end{aligned} \tag{3.14}$$

By (3.13) and (3.14),

$$\begin{aligned}\mathscr{I} &= \mathscr{I}_{ac} + \mathscr{I}_s \\ &< (\beta^{1/2} + \varepsilon_6^{1/2})^2 + (2\|P\|_\infty \varepsilon_7^{1/2} + \|\mathbf{H}\|_\infty \varepsilon_2^{1/2})^2\end{aligned}$$

for $n > \max(n_3, n_4)$. If ε_2, ε_6, ε_7 are small enough, we obtain (3.7) for all $n > \max(n_3, n_4)$.

We have shown that (3.6) and (3.7) hold for some $P_n \in \mathscr{A}$, and so the result follows. ∎

4. Polynomial Approximation: Absolutely Continuous Measures

Notation is as in Section 3. We have seen that (1.5)–(1.8) are independent of the singular components of μ and ν (at least when $\mu_s \perp \nu_s$). The unconstrained infimum $S_\infty(\mu_{ac})$, or, what is the same thing, $T_0(\mu_{ac})$, is computed in Rosenblum and Widom [19]. In this section we apply the abstract theory of Section 2 to compute $S_M(\mu_{ac}, \nu_{ac})$ and $T_\lambda(\mu_{ac}, \nu_{ac})$ for any positive constants M and λ.

We exclude certain trivial cases in order to simplify the discussion.

Case 1. $\mu_{ac} = 0$ or $\nu_{ac} = 0$.

In the former case, $S_M(\mu_{ac}, \nu_{ac}) = T_\lambda(\mu_{ac}, \nu_{ac}) = 0$ for any positive constants M and λ, and in the latter, $S_M(\mu_{ac}, \nu_{ac}) = T_\lambda(\mu_{ac}, \nu_{ac}) = S_\infty(\mu_{ac}) = T_0(\mu_{ac})$ for any positive constants M and λ.

Case 2. $h \perp \mathscr{P}$ in $L^2(\mu_{ac})$.

In this case, $S_M(\mu_{ac}, \nu_{ac}) = T_\lambda(\mu_{ac}, \nu_{ac}) = \int |h|^2 \, d\mu_{ac}$ for any positive constants M and λ.

The discussion is also simplified if polynomials are replaced by Hardy class functions. Recall from (3.1) that $d\mu_{ac} = u d\sigma$ and $d\nu_{ac} = v d\sigma$. We assert that for any positive constants M and λ,

$$S_M(\mu_{ac}, \nu_{ac}) = \inf\left\{\int |h-k|^2 \, u d\sigma \colon k \in \mathbf{H}^2, \int |k|^2 (u+v)\, d\sigma < \infty, \int |k|^2 \, v d\sigma \leqslant M^2\right\}, \tag{4.1}$$

$$S_\infty(\mu_{ac}) = \inf\left\{\int |h-k|^2 \, u d\sigma \colon k \in \mathbf{H}^2, \int |k|^2 \, v d\sigma < \infty\right\} \tag{4.2}$$

and

$$T_\lambda(\mu_{ac}, \nu_{ac}) = \inf\left\{\int |h-k|^2\, u d\sigma + \lambda \int |k|^2\, v d\sigma:\right.$$

$$\left. k \in \mathbf{H}^2, \int |k|^2\, (u+v)\, d\sigma < \infty\right\}, \tag{4.3}$$

$$T_0(\mu_{ac}) = \inf\left\{\int |h-k|^2\, u d\sigma: k \in \mathbf{H}^2, \int |k|^2\, v d\sigma < \infty\right\}. \tag{4.4}$$

The advantage of (4.1)–(4.4) over (1.5)–(1.8) is that $\mathbf{H}^2$ is a Hilbert space whereas $\mathscr{P}$ is not. This is crucial for applying the results of Section 2. The formulas (4.1)–(4.4) are immediate consequences of

4.1. Lemma. *Suppose that k belongs to $\mathbf{H}^2$ on the circle Γ and $\int |k|^2\, w d\sigma < \infty$, where $0 \leqslant w \in L^1(\sigma)$. Then there exist polynomials $\{p_n\}_1^\infty$ such that*

$$\lim_{n \to \infty} \int |k - p_n|^2\, w d\sigma = 0. \tag{4.5}$$

Proof. By replacing w by $w+1$, we can assume that $\log w \in L^1(\sigma)$. Then $w = |g|^2$ σ – a.e. for some outer function $g \in \mathbf{H}^2$. We have $kg \in \mathbf{H}^2$ because $kg \in \mathbf{H}^1$ and

$$\int |kg|^2\, d\sigma = \int |k|^2\, w d\sigma < \infty.$$

Therefore by Beurling's theorem (Duren [6]) there exist polynomials $\{p_n\}_1^\infty$ such that $p_n g \to kg$ in the metric of $\mathbf{H}^2$. Since $w = |g|^2$ σ – a.e., this implies (4.5). ∎

4.2. Theorem. *Exclude Cases* 1 *and* 2 *above and assume further that* $u + v \geqslant \delta$ σ – a.e. *for some $\delta > 0$. For any $t \in (0, \infty)$ define functions c_t and f_t on D by*

$$c_t(z) := \exp\left(\frac{1}{2}\int \frac{\zeta + z}{\zeta - z} \log[u(\zeta) + tv(\zeta)]\, d\sigma(\zeta)\right), \tag{4.6}$$

$$f_t(z) := c_t(z)^{-1} \int \frac{h(\zeta)\, \bar{c}_t(\zeta)^{-1}\, u(\zeta)}{1 - \bar{\zeta} z}\, d\sigma(\zeta). \tag{4.7}$$

Then for any $t \in (0, \infty)$, $c_t, f_t \in \mathbf{H}^2$ *and* $1/c_t \in \mathbf{H}^\infty$. *Further define*

$$m(t) := \int |f_t|^2 \, v d\sigma, \qquad 0 < t < \infty,$$

$$m(0) := \sup_{0 < t < \infty} m(t) (\leqslant \infty).$$

Then $m(\cdot)$ *is a finite-valued, positive, strictly decreasing, continuous function on* $(0, \infty)$ *with limit* 0 *at* ∞. *Let* M *be a positive constant.*

(i) *If* $M^2 \geqslant m(0)$, *then* $S_M(\mu_{ac}, \nu_{ac}) = S_\infty(\mu_{ac})$.

(ii) *If* $M^2 < m(0)$, *then* $S_M(\mu_{ac}, \nu_{ac}) > S_\infty(\mu_{ac})$, *and* $M^2 = m(\lambda)$ *for a unique* $\lambda \in (0, \infty)$. *Moreover,*

$$\lambda M^2 + S_M(\mu_{ac}, \nu_{ac}) = T_\lambda(\mu_{ac}, \nu_{ac}),$$

and $S_M(\mu_{ac}, \nu_{ac})$ *and* $T_\lambda(\mu_{ac}, \nu_{ac})$ *each have the unique extremal function* f_λ.

It should be noted that it is only in the forms (4.1) and (4.3) that the infima $S_M(\mu_{ac}, \nu_{ac})$ and $T_\lambda(\mu_{ac}, \nu_{ac})$ are attained. In the original forms (1.5) and (1.7) they are not attained, in general, even when $\mu = \mu_{ac}$ and $\nu = \nu_{ac}$.

Proof. For each $t \in (0, \infty)$, c_t is the unique outer function such that $c_t(0) > 0$ and

$$|c_t(\zeta)|^2 = u(\zeta) + tv(\zeta), \qquad \sigma-\text{a.e. on } \Gamma. \tag{4.8}$$

It follows easily that $c_t, f_t \in \mathbf{H}^2$ and $1/c_t \in \mathbf{H}^\infty$. Set

$$\mathbf{K} = \mathbf{H}^2, \qquad \mathbf{H}_1 = L^2(u d\sigma), \qquad \mathbf{H}_2 = L^2(v d\sigma).$$

Let A, B be the inclusion mappings from $\mathbf{K}$ to $\mathbf{H}_1$, $\mathbf{H}_2$, respectively. Thus $D(A)$, $D(B)$, and $\mathscr{D} := D(A) \cap D(B)$ consist of all $k \in \mathbf{H}^2$ for which

$$\int |k|^2 \, u d\sigma, \qquad \int |k|^2 \, v d\sigma, \qquad \text{and} \qquad \int |k|^2 \, (u + v) \, d\sigma,$$

respectively, are finite. In the notation of Section 2, the formulas (4.1)–(4.4) take the form

$$S_M(\mu_{ac}, \nu_{ac}) = I_M, \qquad S_\infty(\mu_{ac}) = I_\infty,$$

$$T_\lambda(\mu_{ac}, \nu_{ac}) = J_\lambda, \qquad T_0(\mu_{ac}) = J_0.$$

The conditions (C1)–(C4) of Section 2 are readily verified from our assumptions. We check only (C4). It is to be shown that there is no $d \in \mathscr{D}$ such that $Ad = h_A$ in $L^2(u d\sigma)$ and $Bd = 0$ in $L^2(v d\sigma)$. Argue by contradiction. If d is such a function, then $|d|^2 \, v = 0$ σ-a.e. on Γ. We have excluded the possibility that $v=0$ σ-a.e. (Case 1 at the beginning of Section 4), and so $d=0$ on a set of positive measure. By the F. and M. Riesz theorem, $d=0$ σ-a.e. Hence $h_A=Ad=0$ in $L^2(ud\sigma)$. However, this possibility is also excluded (Case 2 at the beginning of Section 4). Therefore no such d exists, and (C4) holds.

We will show that for each $t\in(0,\infty)$, $(AT_t^{-1})^*h$ coincides with the function f_t defined by (4.7). Once this is known, the remaining assertions of Theorem 4.2 follow as a special case of Theorem 2.6.

Claim. For each $t\in(0,\infty)$ and $\alpha\in D$,

$$T_t^{-1}\colon 1/(1-\bar{\alpha}\zeta)\to \bar{c}_t(\alpha)^{-1}\,c_t(\zeta)^{-1}/(1-\bar{\alpha}\zeta). \tag{4.9}$$

To see this, fix $\alpha\in D$ and set

$$g_t(\zeta)=\bar{c}_\lambda(\alpha)^{-1}\,c_t(\zeta)^{-1}/(1-\bar{\alpha}\zeta). \tag{4.10}$$

To prove (4.9) is sufficient to show that g_t is in the domain of $T_t=\mathbf{C}_t^*\mathbf{C}_t$ (this is the definition of T_t; see (2.5)) and

$$T_t\colon g_t(\zeta)\to 1/(1-\bar{\alpha}\zeta).$$

Now $g_t\in\mathbf{H}^\infty$ and so $g_t\in\mathscr{D}=D(\mathbf{C}_t)$. It is therefore enough to show that

$$\langle \mathbf{C}_t k, \mathbf{C}_t g\rangle=\langle k(\zeta), 1/(1-\bar{\alpha}\zeta)\rangle \tag{4.11}$$

for every $k\in D(\mathbf{C}_t)=\mathscr{D}$. The identity (4.11) is equivalent to

$$\int k\bar{g}_t u\,d\sigma+t\int k\bar{g}_t v\,d\sigma=f(\alpha).$$

By (4.8) and (4.9), this reduces to

$$c_t(\alpha)^{-1}\int\frac{k(\zeta)\,c_t(\zeta)}{1-\alpha\bar{\zeta}}\,d\sigma(\zeta)=f(\alpha),$$

which holds by Cauchy's theorem for $\mathbf{H}^1$. The claim follows.

Finally, for any $t\in(0,\infty)$, by (4.9) we have

$$\begin{aligned}((AT_t^{-1})^*h)(\alpha)&=\langle h(\zeta), AT_t^{-1}\{1/(1-\bar{\alpha}\zeta)\}\rangle\\&=c_t(\alpha)^{-1}\int\frac{h(\zeta)\,\bar{c}_t(\zeta)^{-1}\,u(\zeta)}{1-\bar{\zeta}\alpha}\,d\sigma(\zeta)\\&=f_t(\alpha)\end{aligned}$$

for all $\alpha\in D$. Thus $(AT_t^{-1})^*h=f_t$, and as noted above the remaining assertions of Theorem 4.2 follow from Theorem 2.6. ∎

Remark. The operator T_t that appears in the proof of Theorem 4.2 is a Toeplitz operator (in general, unbounded). Formula (4.9) is equivalent to the generating formula for resolvents [3, 10, 16, 18]. An alternative proof of (4.9) can be given along these lines, but when everything is considered, it is much more complicated in the unbounded case than the argument given above. In the bounded case, the proof of (4.9) via the generating formula for resolvents is prabably more transparent than the argument given above.

5. Constrained Forms of Szegö's Infimum

Let μ be a nonnegative, finite Borel measure on Γ with Lebesgue decomposition as in (3.1). For any positive constants M and λ set

$$\hat{S}_M(\mu) := \inf\left\{\int |\bar{\zeta}-p|^2\, d\mu : p \in \mathscr{P}, \int |p|^2\, d\sigma \leqslant M^2\right\}, \tag{5.1}$$

$$\hat{S}_\infty(\mu) := \inf\left\{\int |\bar{\zeta}-p|^2\, d\mu : p \in \mathscr{P}\right\} \tag{5.2}$$

and

$$\hat{T}_\lambda(\mu) := \inf\left\{\int |\bar{\zeta}-p|^2\, d\mu + \lambda \int |p|^2\, d\sigma : p \in \mathscr{P}\right\}, \tag{5.3}$$

$$\hat{T}_0(\mu) := \inf\left\{\int |\bar{\zeta}-p|^2\, d\mu : p \in \mathscr{P}\right\}. \tag{5.4}$$

By the classical Szegö infimum (Grenander and Szegö [8]),

$$\hat{S}_\infty(\mu) = \hat{T}_0(\mu) = \begin{cases} \exp(\int \log u\, d\sigma), & \log u \in L^1(\sigma), \\ 0, & \log u \notin L^1(\sigma). \end{cases}$$

The infima (5.1)–(5.4) correspond to (1.5)–(1.8) with $h(\zeta) = \bar{\zeta}$ and $v = \sigma$. The results we state here strengthen and extend those of [1] on Sezgö's infimum with constraints.

5.1. Theorem. *For any positive constants M and λ,*

$$\hat{S}_M(\mu) = \hat{S}_M(\mu_{ac}) \qquad \text{and} \qquad \hat{T}_M(\mu) = \hat{T}_M(\mu_{ac}).$$

5.2. Theorem. *Assume that μ_{ac} is not identically 0. For any $t \in (0, \infty)$, define*

$$c_t(z) := \exp\left(\frac{1}{2}\int \frac{\zeta + z}{\zeta - z} \log[u(\zeta) + t]\, d\sigma(\zeta)\right), \tag{5.5}$$

$$f_t(z) := z^{-1}[1 - c_t(0)/c_t(z)] \tag{5.6}$$

on D, and set

$$m(t) := \int (u+t)^{-1}\, d\sigma \exp\left(\int \log(u+t)\, d\sigma\right) - 1,$$
$$m(0) := \sup_{0<t<1} m(t) (\leqslant \infty). \tag{5.7}$$

Then $m(\cdot)$ *is a positive, strictly decreasing, continuous function on* $(0, \infty)$ *with limit* 0 *at* ∞. *Let* M *and* λ *be positive constants.*

(i) *If* $M^2 \geqslant m(0)$, *then* $\hat{S}_M(\mu_{ac}) = \hat{S}_\infty(\mu_{ac})$.

(ii) *If* $M^2 = m(\lambda)$, *then*

$$\lambda M^2 + \hat{S}_M(\mu_{ac}) = \hat{T}_\lambda(\mu_{ac}) = \exp\left(\int \log(u+\lambda)\, d\sigma\right) - \lambda, \tag{5.8}$$

and $\hat{S}_M(\mu_{ac})$ *and* $\hat{T}_\lambda(\mu_{ac})$ *each have the unique extremal* f_λ.

Proofs. Theorems 5.1 and 5.2 are particular cases of Theorems 3.1 and 4.2. We leave the straightforward calculations to the reader. ∎

Remark. As in [1], a little more can be said in Theorem 5.2. Namely, $m(0) < \infty$ if and only if $1/u \in L^1(\sigma)$. In this case, assertion (ii) is true as stated for $\lambda = 0$ if c_0 and f_0 are defined by (5.5) and (5.6) with $t = 0$. We omit the proofs of these assertions.

5.3. Example. We have $\hat{S}_M(\mu) \downarrow \hat{S}_\infty(\mu)$ as $M \to \infty$ and $\hat{T}_\lambda(\mu) \downarrow \hat{T}_0(\mu)$ as $\lambda \downarrow 0$. It is natural to ask for the rates of convergence. We are unable to determine this in general, but we compute an example that may be instructive.

Let $d\mu = u\,d\sigma$, where $u = \chi_E$ is the characteristic function of a Borel set $E \subseteq \Gamma$ with $0 < |E| < 1$ ($|E| = \sigma(E)$). Then $\hat{S}_\infty(\mu) = \hat{T}_0(\mu) = 0$, so $\hat{S}_M(\mu) \downarrow 0$ as $M \to \infty$, and $\hat{T}_M(\mu) \downarrow 0$ as $\lambda \downarrow 0$. It turns out that the rates of convergence depend only on $|E|$. We state this result in slightly more general form as follows.

Proposition. *Let* $d\mu = u\,d\sigma$, *where* $u = 0$ $\sigma -$ a.e. *on a Borel set* $E \subseteq \Gamma$ *such that* $0 < |E| < 1$ *and* $u \neq 0$ $\sigma -$ a.e. *on* $\Delta := \Gamma \setminus E$, *with* $\int_\Delta \log u\,d\sigma > -\infty$. *Then*

$$\hat{S}_M(\mu) = \frac{\mathbf{C}}{M^{2|E|/(1-|E|)}}(1 + \mathcal{O}(1)) \qquad \textit{as} \quad M \to \infty, \tag{5.9}$$

$$\hat{T}_\lambda(\mu) = D\lambda^{|E|}(1 + \mathcal{O}(1)) \qquad \textit{as} \quad \lambda \downarrow 0, \tag{5.10}$$

where $\mathbf{C}$ *and* D *are positive constants depending only on* $|E|$ *and* $\int_\Delta \log u\,d\sigma$.

Proof. We apply Theorem 5.2. We are in the case $m(0)=\infty$. Let M and λ be related by $M^2=m(\lambda)$, so $M\to\infty$ corresponds to $\lambda\downarrow 0$. For $0<\lambda<\infty$, by (5.7),

$$\begin{aligned} m(\lambda) &= \int (u+\lambda)^{-1}\,d\sigma \exp\left(\int \log(u+\lambda)\,d\sigma\right)-1 \\ &= \left(\int_E \lambda^{-1}\,d\sigma + \int_\Delta (u+\lambda)^{-1}\,d\sigma\right)\exp\left(\left(\int_E+\int_\Delta\right)\log(u+\lambda)\,d\sigma\right)-1 \\ &= \left(|E|\,\lambda^{-1}+\int_\Delta (u+\lambda)^{-1}\,d\sigma\right)\lambda^{|E|}\exp\left(\int_\Delta \log(u+\lambda)\,d\sigma\right)-1 \\ &= |E|\,\lambda^{-1+|E|}\left(1+|E|^{-1}\int_\Delta \lambda(u+\lambda)^{-1}\,d\sigma\right) \\ &\quad \times\exp\left(\int_\Delta \log(u+\lambda)\,d\sigma\right)-1 \\ &= |E|\,\lambda^{-1+|E|}\exp\left(\int_\Delta \log u\,d\sigma\right)(1+\mathcal{O}(1)) \end{aligned}$$

as $\lambda\downarrow 0$. Since $M^2=m(\lambda)$, this yields

$$\lambda=\left(|E|\,M^{-2}\exp\left(\int_\Delta \log u\,d\sigma\right)\right)^{1/(1-|E|)}(1+\mathcal{O}(1)). \tag{5.11}$$

By (5.8),

$$\begin{aligned} \hat{S}_M(\mu) &= \int u(u+\lambda)^{-1}\,d\sigma \exp\left(\int \log(u+\lambda)\,d\sigma\right) \\ &= \int_\Delta u(u+\lambda)^{-1}\,d\sigma \exp\left(\left(\int_E+\int_\Delta\right)\log(u+\lambda)\,d\sigma\right) \\ &= |\Delta|\,\lambda^{|E|}\exp\left(\int_\Delta \log u\,d\sigma\right)(1+\mathcal{O}(1)) \end{aligned}$$

as $M\to\infty$. Substituting (5.11) into this last expression, we obtain (5.9) with

$$\mathbf{C}=(1-|E|)|E|^{|E|/(1-|E|)}\exp\left(\left(1+\frac{|E|}{1-|E|}\right)\int_\Delta \log u\,d\sigma\right).$$

Again by (5.8) and what we have just shown,

$$\begin{aligned}\hat{T}_\lambda(\mu) &= \hat{S}_M(\mu) + \lambda m(\lambda) \\ &= |\Delta|\ \lambda^{|E|} \exp\left(\int_\Delta \log u d\sigma\right)(1+\mathcal{O}(1)) \\ &\quad + |E|\ \lambda^{|E|} \exp\left(\int_\Delta \log u d\sigma\right)(1+\mathcal{O}(1)) \\ &= \exp\left(\int_\Delta \log u d\sigma\right)\lambda^{|E|}(1+\mathcal{O}(1))\end{aligned}$$

as $\lambda \downarrow 0$. Hence (5.10) holds with $D = \exp(\int_\Delta \log u d\sigma)$. ∎

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